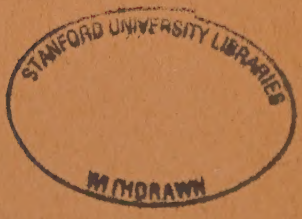
SOME PROPERTIES OF THE EXPONENTIAL MEAN

A THESIS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL OF
CORNELL UNIVERSITY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

BY

JULIA DALE



ITHACA, NEW YORK,
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Some Properties of the Exponential Mean.

By Julia Dale.

1. Introduction.

Among the many definitions that have been given for evaluating a divergent series, the exponential mean is frequently one of the most convenient. In this paper, the purpose has been to study some of the properties of this definition and show its relation to Euler's Summation Process.

Unless otherwise stated, we use the symbol

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots,$$

and define

$$x_n = u_1 + u_2 + u_3 + \cdots + u_n.$$

The exponential mean * assigns to a divergent series the value $\lim_{n\to\infty} y_n$, if this limit exists, where

$$(E_r) \qquad y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} x_k.$$

The successive coefficients are the terms of the binomial expansion of

$$[(1-1/r) + 1/r]^{n-1}.$$

The transformation which we have designated in this paper as Euler's Summation Process was defined by Euler,† when applied to a series of decreasing terms of alternating signs, in the following manner:

$$S = s_{n-1} + (-1)^{n-1}[\tfrac{1}{2}u_n + \tfrac{1}{4}\Delta^1 u_n + \tfrac{1}{8}\Delta^2 u_n + \cdots + \tfrac{1}{2^k}\Delta^{k-1}u_n].$$

where n is an integer and

$$s_{n-1} = u_1 - u_2 + u_3 + \cdots + (-1)^{n-1}u_{n-1}$$

and

$$\Delta^1 u_n = u_n - u_{n+1},$$

$$\Delta^2 u_n = \Delta^1 u_n - \Delta^1 u_{n+1} = u_n - 2u_{n+1} + u_{n+2},$$

. . . . . .

* Hurwitz: "Report on topics in the theory of divergent series," *Bulletin of the American Mathematical Society*, Vol. 28 (1922), p. 17. This paper will be referred to as *Report*.

Also, Hausdorff: "Summationsmethoden und Momentfolgen," *Mathematische Zeitschrift*, Vol. 9 (1921), p. 86.

† Euler: *Institutiones Calculi differentialis*, Part II, De transformatione serierum, p. 232.

Let us use this method to evaluate the series *

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} \cdots .$$

The sum of the first eight terms to six decimals is 0.634524. Applying the transformation to the next seven terms, we obtain

$$\begin{aligned} S &= .634524 + \tfrac{1}{2}(.111111) + \tfrac{1}{4}(.011111) + \tfrac{1}{8}(.002020) + \\ &\quad 1/16(.000505) + 1/32(.000156) + 1/64(.000057) \\ &= .634524 + .058624 \\ &= .693148. \end{aligned}$$

The actual sum of the series is $\log 2 = .69314718$.

"To reach this degree of accuracy we should have to use over a hundred thousand terms of the original series." *

The first discussion of the accuracy of this transformation is due to Poncelet.† He applies this transformation to slowly convergent series of alternating signs and computes a formula for the limit of error, when we stop with the nth term. Poncelet, also, defines a generalized form of this transformation. It is designated in this paper as Euler's Generalized Summation Process. Kummer,‡ Markoff,§ and Catalan ‖ have worked with different forms of this transformation, principally with the original form. Ames ¶ has computed a formula for the limit of error, when the original form of Euler's Summation Process is applied to slowly convergent series of certain types. He has also suggested that, if this transformation be applied to certain types of divergent series, it renders them convergent.

In this paper we have applied this transformation in its original form and in the generalized form to divergent series and studied some of its properties. In parts 2 and 3 it is proved that not only the original Euler's Summation Process but also the generalized one is a regular ** definition.

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* Bromwich: *Theory of Infinite Series*, (1908), p. 58.

† Poncelet: "Application de la méthode des Moyennes à la transformation, au calcul numérique et à la détermination des limites du reste des séries," *Journal für Mathematik von Crelle*, Vol. 13 (1833), p. 1.

‡ Kummer: "Eine neue Methode, die Numerischen Summen langsam convergierender Reihen zu berchnen," *Journal für Mathematik von Crelle*, Vol. 16 (1837), p. 206.

§ Markoff: *Differenzenrechnung*, Leipzig, (1896).

‖ Catalan: "Mémoire sur la Transformation des Séries," *Mémoires Courronés de l'Académie de Belgique*, Vol. 33 (1867), p. 2.

¶ Ames: "Evaluation of Slowly Convergent Series," *Annals of Mathematics*, Series 2, Vol. 3(1902), p. 185.

** The terminology here used is that of Professor Hurwitz, *Report*.

A definition is said to be *regular,* if it evaluates every convergent sequence, giving it the value to which it converges. Also, it is proved that this definition is equivalent * to the exponential mean. Two definitions are defined as being *equivalent* when each evaluates to the value l every sequence evaluated by the other to the value l.

Part 4 is devoted to the application of the exponential mean to the Cauchy product of two series. Knopp ‡ built a definition by applying E_2 successively p times to the given series. He investigated some of the properties of this definition, among them the value assigned, if one exists, to the Cauchy product of two series. He proved some theorems that are generalizations of theorems of Abel and Mertens for convergent series. Here we give an explicit definition of summability which evaluates the Cauchy product of two series summable respectively E_r and E_s.

2. The Euler Summation Process.

We consider the Euler process as applied directly from the first term, that is with $n = 1$ in the formula given in the Introduction.

For the series

$$u_1 - u_2 + u_3 - u_4 + \cdots,$$

define

1) $$s_n = \sum_{k=1}^{n} \tfrac{1}{2}^k \Delta^{k-1} u_1,$$

where

$$\Delta^0 u_1 = u_1,$$
$$\Delta^1 u_1 = u_1 - u_2,$$
$$\Delta^2 u_1 = \Delta^1 u_1 - \Delta^1 u_2 = u_1 - 2u_2 + u_3,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta^k u_1 = \Delta^{k-1} u_1 - \Delta^{k-1} u_2$$
$$= u_1 - k u_2 + \frac{k(k-1)}{1 \cdot 2} u_3 - \cdots + (-1)^k u_{k+1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

then the generalized value of the series is to be

$$s = \lim_{n \to \infty} s_n.$$

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* The terminology here used is that of Professor Hurwitz, *Report.*

‡ Knopp: "Über das Eulersche Summierungsverfahren," *Mathematische Zeitschrift,* Vol. 15 (1922-23), p. 226; Vol. 18 (1923), p. 125.

In order that the definition may be applied to a series written in the usual way with positive signs, we define

$$v_n = (-1)^{n-1}u_n.$$

We have

$$s_n = \sum_{k=1}^{n} \tfrac{1}{2}{}^k[u_1 - (k-1)u_2 + \frac{(k-1)(k-2)}{1\cdot 2}u_3 - \cdots + (-1)^{k-1}u_k],$$

or, by inversion of the order of summation,

$$2) \qquad s_n = \sum_{k=1}^{n}\sum_{h=1}^{n-k+1} \frac{(-1)^{k-1}}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}u_k.$$

Thus the relation of s_n to v_n is given by

$$(\mathcal{E}_2) \qquad s_n = \sum_{k=1}^{n} a_{n,k}v_k,$$

where

$$3) \qquad a_{n,k} = \sum_{h=1}^{n-k+1} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}.$$

THEOREM I. *$\mathcal{E}_2$ is a regular transformation.*

In order that a transformation be regular, when applied directly to the terms of the series, it is necessary and sufficient that *

a) for each k, $\lim_{n\to\infty} a_{n,k} = 1$

b) for all n $\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} |$ is bounded.

Obviously, condition a) is satisfied, since

$$\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \sum_{h=1}^{n-k+1} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}$$

$$= \sum_{h=1}^{\infty} \frac{1}{2^{h+k-1}} \frac{(h+k-2)!}{(h-1)!(k-1)!}$$

$$= 1/2^k\,[1-\tfrac{1}{2}]^{-k} = 1.$$

In order to establish condition b), we write 3) in the form

$$a_{n,k} = 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}.$$

* Carmichael: "General Aspects of the Theory of Summable Series," *Bulletin of the American Mathematical Society*, Vol. 25 (1916-1919), p. 97.

Then

$$4) \quad a_{n,k} - a_{n,k+1} = 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}$$

$$- \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{h!(k-1)!}$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{(h-1)!(k-1)!} \left[\frac{1}{h} - \frac{1}{k}\right]$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k-1)!}{(h-1)!(k-1)!} \left[\frac{1}{h} - \frac{1}{k} + \frac{2}{k}\right]$$

$$- \sum_{h=1}^{n-k} \frac{1}{2^{h+k-1}} \frac{(h+k-1)!}{(h-1)!\,k!}$$

$$= 1/2^k + \sum_{h=1}^{n-k} \frac{1}{2^{h+k}} \frac{(h+k)!}{h!\,k!}$$

$$- \sum_{h=0}^{n-k-1} \frac{1}{2^{h+k}} \frac{(h+k)!}{h!\,k!}$$

$$= \frac{1}{2^n} \frac{n!}{(n-k)!\,k!} .$$

From this, we see that

$$\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} | = \frac{1}{2^n} [(1+1)^n - 2] = 1 - \frac{1}{2^{n-1}} < 1$$

Then

$\sum_{k=1}^{n-1} | a_{n,k} - a_{n,k+1} |$ is bounded for all n and $\mathcal{E}_2$ satisfies the conditions for regularity.

We shall now express this transformation in the form of a sequence-to-sequence transformation. Form the sequence

$$5) \quad x_1, x_2, x_3, x_4, \cdots$$

where $x_k = v_1 + v_2 + \cdots + v_k$.

then $v_k = x_k - x_{k-1}$

and

$$(\mathcal{E}_2) \quad s_n = \sum_{k=1}^{n} a_{n,k}(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} a_{n,k} x_k - \sum_{k=0}^{n-1} a_{n,k+1} x_k$$

$$= \sum_{k=1}^{n-1} (a_{n,k} - a_{n,k+1}) x_k + a_{n,n} x_n .$$

Thus

$$s_n = \sum_{k=1}^{n} b_{n,k} x_k,$$

where

$$b_{n,k} = \begin{cases} a_{n,k} - a_{n,k+1} & k < n, \\ a_{n,n}, & k = n. \end{cases}$$

Furthermore, by 4), we find

$$b_{n,k} = \frac{1}{2^n} \cdot \frac{n!}{(n-k)!\,k!}$$

THEOREM II. *$\mathcal{E}_2$ is equivalent to the exponential mean E_2.*

In this work the notation $A \sim B$ will be used to indicate that two given sequences A and B are equivalent.

Define

(P) $$x'_k = \begin{cases} x_{k-1} & k > 1, \\ 0 & k = 1. \end{cases}$$

It is readily verified that

6) $$E_2 P = P \mathcal{E}_2.$$

The transformation $\mathcal{E}_2$ has been shown to be regular, therefore it is obvious that

7) $$\mathcal{E}_2 \sim P \mathcal{E}_2.$$

If a sequence is summable E_r, then the sequence obtained by prefixing an element is summable to the same value,* therefore

8) $$E_2 \sim E_2 P.$$

By means of 6), 7), 8), we see that

$$E_2 \sim \mathcal{E}_2.$$

3. THE GENERALIZED EULER SUMMATION PROCESS.

For the generalized Euler Summation Process, as applied to the series

$$u_1 t - u_2 t^2 + u_3 t^3 - u_4 t^4 + \cdots,$$

we define

9) $$s_n = \sum_{k=1}^{n} (t/1+t)^k \Delta^{k-1} u_1,$$

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* Hurwitz, *Report.*

where

$$\Delta^0 u_1 = u_1,$$
$$\Delta^1 u_1 = u_1 - u_2,$$
$$\Delta^2 u_1 = \Delta^1 u_1 - \Delta^1 u_2 = u_1 - 2u_2 + u_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\Delta^k u_1 = \Delta^{k-1} u_1 - \Delta^{k-1} u_2 = u_1 - k u_2 + \frac{k(k-1)}{1 \cdot 2} u_3 - \cdots$$
$$\cdots + (-1)^k u_{k+1}$$

then the generalized value of the series is to be given by

$$s = \lim_{n \to \infty} s_n.$$

In order to find an expression for s_n, involving directly the terms of the series, we define

$$v_n = (-1)^{n-1} u_n t^n.$$

We have

$$s_n = \sum_{k=1}^{n} (t/1+t)^k \left[u - \frac{(k-1)}{1!} u_2 + \frac{(k-1)(k-2)}{1 \cdot 2} u_3 \cdots + (-1)^{k-1} u_k \right].$$

or, by inversion of the order of summation,

10) $$s_n = \sum_{k=1}^{n} \sum_{h=1}^{n-k+1} \frac{(-1)^{k+1}(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}} u_k t^k.$$

Thus the relation of s_n to v_n is given by

$$(\mathcal{E}_{1+t}) \quad s_n = \sum_{k=1}^{n} a_{n,k} v_k,$$

where

11) $$a_{n,k} = \sum_{h=1}^{n-k+1} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}.$$

THEOREM III. *$\mathcal{E}_{1+t}$, where $t > 0$, is a regular transformation.*

Returning to the conditions for regularity as stated in part 2, we find condition a) satisfied, since

$$\lim_{n \to \infty} a_{n,k} = \lim_{n \to \infty} \sum_{h=1}^{n-k+1} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}$$
$$= \sum_{h=1}^{\infty} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}}$$
$$= \frac{1}{(1+t)^k} \left[1 - \frac{t}{1+t} \right]^{-k} = 1.$$

In order to establish condition b) we write

$$12) \qquad a_{n,k} - a_{n,k+1} = \sum_{h=1}^{n-k+1} \frac{(h+k-2)!}{(h-1)!(k-1)!} \frac{t^{h-1}}{(1+t)^{h+k-1}} - \sum_{h=1}^{n-k} \frac{(h+k-1)!}{(h-1)!\,k!} \frac{t^{h-1}}{(1+t)^{h+k}}.$$

Changing the variable of summation in the first sum, we have

$$a_{n,k} - a_{n,k+1} = \sum_{h=0}^{n-k} \frac{(h+k-1)!}{h!(k-1)!} \frac{t^h}{(1+t)^{h+k}} - \sum_{h=1}^{n-k} \frac{(h+k-1)!}{(h-1)!\,k!} \frac{t^{h-1}}{(1+t)^{h+k}}$$

$$= \frac{1}{(1+t)^k} + \sum_{h=1}^{n-k} \frac{(h+k-1)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} \left[k - \frac{h}{t}\right]$$

$$= \frac{1}{(1+t)^k} + \sum_{h=1}^{n-k} \frac{(h+k-1)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} \left[k - \frac{h}{t} + \frac{h}{t} + h\right] - \sum_{h=1}^{n-k} \frac{(h+k-1)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} \frac{h(1+t)}{t}$$

$$= \frac{1}{(1+t)^k} + \sum_{h=1}^{n-k} \frac{(h+k)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} - \sum_{h=1}^{n-k} \frac{(h+k-1)!}{(h-1)!\,k!} \frac{t^{h-1}}{(1+t)^{h+k-1}}.$$

Changing the variable of summation in the second sum, we have

$$a_{n,k} - a_{n,k+1} = \frac{1}{(1+t)^k} + \sum_{h=1}^{n-k} \frac{(h+k)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}} - \sum_{h=0}^{n-k-1} \frac{(h+k)!}{h!\,k!} \frac{t^h}{(1+t)^{h+k}}$$

$$= \frac{t^{n-k}}{(1+t)^n} \frac{n!}{(n-k)!\,k!}$$

Hence, we see that

$$\sum_{k=1}^{n-1} |a_{n,k} - a_{n,k+1}| = (t/1+t)^n [(1+1/t)^n - 1 - 1/t^n]$$

$$= 1 - (t/1+t)^n - 1/(1+t)^n < 1.$$

Consequently, condition b) is satisfied.*

Let us express the generalized Euler summation transformation in the

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* We may choose to consider $\mathcal{E}_{1+t}$ with coefficients given by 20) for any value of t, with the convention that $\mathcal{E}_1$ is the identity. The method of the text can readily be extended to show that $\mathcal{E}_{1+t}$ is regular if and only if t is real and ≥ 0. Furthermore, for these values $\mathcal{E}_{1+t}$ is totally regular, in the sense of Professor Hurwitz's *Report*, p. 29.

form of a sequence-to-sequence transformation. As in the preceding case, we find,

$$s_n = \sum_{k=1}^{n-1} (a_{n,k} - a_{n,k+1}) x_k + a_{n,n} x_n,$$

Thus

$$s_n = \sum_{k=1}^{n} \beta_{n,k} x_k,$$

where

$$\beta_{n,k} = \frac{t^{n-k}}{(1+t)^n} \quad \frac{n!}{(n-k)!\,k!}$$

THEOREM IV. *$\mathcal{E}_{1+t}$, for $t > 0$, is equivalent to the exponential mean E_{1+t}.*

As before we can show that

13) $$E_{1+t} P = P\, \mathcal{E}_{1+t}.$$

The transformation $\mathcal{E}_{1+t}$ has been shown to be regular, then it is obvious that

14) $$\mathcal{E}_{1+t} \sim P\, \mathcal{E}_{1+t}$$

as also *

15) $$E_{1+t} \sim E_{1+t} P.$$

It follows from 13), 14), 15) that

$$E_{1+t} \sim \mathcal{E}_{1+t}.$$

THEOREM V. *$\mathcal{E}_{1+t}\ \mathcal{E}_{1+s} = \mathcal{E}_{(1+t)(1+s)}$, for $t > 0$ and $s > 0$.*

Let us apply $\mathcal{E}_{(1+t)(1+s)}$ to 5), then we have

16) $$w_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!\,k!} \quad \frac{(t+s+ts)^{n-k}}{(1+t)^n (1+s)^n}\ x_k.$$

Applying $\mathcal{E}_{1+t}$ to sequence 5) and then $\mathcal{E}_{1+s}$, we find

$$y_n = \sum_{h=1}^{n} \frac{n!}{(n-h)!\,h!} \quad \frac{t^{n-h}}{(1+t)^n} \sum_{k=1}^{h} \frac{h!}{(h-k)!\,k!} \quad \frac{s^{h-k}}{(1+s)^h}\ x_k.$$

$$= \sum_{k=1}^{n} (t/1+t)^n x_k \sum_{h=k}^{n} \frac{n!}{(n-h)!\,(h-k)!\,k!} \quad \frac{1}{t^h} \quad \frac{s^{h-k}}{(1+s)^h}$$

$$= \sum_{k=1}^{n} (t/1+t)^n \frac{n!}{(n-k)!\,k!} \quad \frac{1}{t^k} \quad \frac{1}{(1+s)^k} \left[1 + \frac{s}{t(1+s)}\right]^{n-k} x_k.$$

* Hurwitz: *Report*, p. 31.

that is

17) $$y_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!\,k!} \frac{(t+s+ts)^{n-k}}{(1+t)^n(1+s)^n} x_k.$$

Therefore from 16) and 17) we have

$$w_n = y_n.$$

By a successive application of this result, we infer

COROLLARY 1. $\mathcal{E}_{(1+t_1)(1+t_2)\cdots(1+t_n)} = \mathcal{E}_{1+t_1} \mathcal{E}_{1+t_2} \cdots \mathcal{E}_{1+t_n}$.

COROLLARY 2. $\mathcal{E}^p{}_{1+t} = \mathcal{E}_{(1+t)^p}$.

THEOREM VI. *If a sequence is summable $\mathcal{E}_{1+t}$ to the value l, then any sequence obtained by omitting or prefixing a finite number of terms is summable $\mathcal{E}_{1+t}$ to the value l.*

If a sequence is summable E_{1+t} to the value l, then the sequence obtained by omitting or prefixing a term is summable E_{1+t} to the value l.* It was proved Theorem IV, that

$$E_{1+t} \sim \mathcal{E}_{1+t}$$

By means of the definition of equivalence $\mathcal{E}_{1+t}$ enjoys the same property.

It is sometimes convenient to sum by ordinary addition the first p terms of the series

$$u_1 t - u_2 t^2 + u_3 t^3 + \cdots$$

and then apply the generalized Euler Summation Process to the remaining terms. The generalized sum of the series, if it exists, would then be defined as

$$\lim_{n\to\infty} s_n$$

where

$$s_n = s_p + \sum_{k=1}^{n} (t/1+t)^k \Delta^{k-1} u_{p+1},$$

and

$$s_p = u_1 t - u_2 t^2 + \cdots + (-1)^{p-1} u_p t^p.$$

But here again, in order to find an expression for s_n, involving directly the terms of the series, we make the linear transformation

$$v_n = (-1)^{n-1} u_n t^n.$$

Then we have

$(\mathcal{E}_{1+t,p})$ $$s_n = x_p + \sum_{k=1}^{n} a_{n,k} v_{p+k}$$

where $a_{n,k}$ is given by 11).

* Hurwitz: *Report*, p. 31.

THEOREM VII. *$\mathcal{E}_{1+t,p}$ is equivalent to $\mathcal{E}_{1+t}$.*

We can obviously write

$$\mathcal{E}_{1+t,p}(x_n) = x_p + \mathcal{E}_{1+t}(x_{p+n} - x_p)$$
$$= [x_p - \mathcal{E}_{1+t}(x_p)] + \mathcal{E}_{1+t}(x_{p+n}).$$

The expression in the brackets has the limit zero as $n \to \infty$ since $\mathcal{E}_{1+t}$ is regular. Thus if $\mathcal{E}_{1+t,p}$ evaluates (x_n) to l, $\mathcal{E}_{1+t}$ will evaluate (x_{p+n}) to l, and by Theorem VI, $\mathcal{E}_{1+t}$ will evaluate (x_n) to l. Conversely, if $\mathcal{E}_{1+t}$ evaluates (x_n) to l, $\mathcal{E}_{1+t}$ will evaluate (x_{p+n}) to l and $\mathcal{E}_{1+t,p}$ will evaluate (x_n) to l. Hence we may state

$$\mathcal{E}_{1+t,p} \sim \mathcal{E}_{1+t}.$$

4. MULTIPLICATION OF SERIES.

The expressions obtained by multiplying the terms of one series $\sum u_n$ by the terms of another series $\sum u_n$ may be arranged formally as a double series $\sum u_m v_n$. We may collect the terms of this double series into a simple series $\sum w_n$, by means of the Cauchy product, where

$$w_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1.$$

Before taking up the case of the summability of the Cauchy product of two series, which are summable E_r, we prove two lemmas.

Take two sequences

$$(x_n) : x_1, x_2, x_3, \cdots,$$
$$(y_n) : y_1, y_2, y_3, \cdots,$$

and an infinite matrix

$$(a_{n,k}) : \left\| \begin{matrix} a_{11}, & 0, & 0, & 0, & . & . & . & . & . \\ a_{21}, & a_{22}, & 0, & 0, & . & . & . & . & . \\ a_{31}, & a_{32}, & a_{33}, & 0, & 0, & . & . & . & . \\ . & . & . & & & & & & \\ . & . & . & & & & & & \\ . & . & . & & & & & & \\ . & . & . & & & & & & \end{matrix} \right\| .$$

Then the transformation

18) $$z_n = \sum_{k=1}^{n} a_{n,k} x_k y_{n-k+1}$$

defines the sequence

$$(z_n) \; : \; z_1, \; z_2, \; z_3, \; \cdots .$$

We indicate the transformation 18) by $\mathcal{A}$, and the relation by

$$z_n = \mathcal{A}(x_n y_n).$$

LEMMA 1. *The necessary and sufficient conditions that, if* $\lim_{n\to\infty} y_n = y$ *and* $\lim_{n\to\infty} x_n = x$, *then* $\lim_{n\to\infty} z_n = xy$, *where*

$$z_n = \sum_{k=1}^{n} a_{n,k} x_k y_{n-k+1}$$

are

a) $\lim_{n\to\infty} a_{n,k} = 0$ *for each* k;

b) $\lim_{n\to\infty} a_{n,n-k+1} = 0,$ *for each* k;

c) $\sum_{k=1}^{n} | a_{n,k} | < c,$ *for all* n;

d) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1.$

We shall first establish the necessity of these conditions.

Take

$$y_1 = y_2 = y_n = \cdots = 1$$

and leave (x_n) arbitrary. Substiuting this value of (y_n) in 18), we have

$$z_n = \sum_{k=1}^{n} a_{n,k} x_k.$$

This is now an ordinary sequence-to-sequence transformation. For a transformation of this type to be regular, it is necessary and sufficient that *

a′) $\lim_{n\to\infty} a_{n,k} = 0,$ for each k

b′) $\sum_{k=1}^{n} | a_{n,k} | < c,$ for all n,

c′) $\lim_{n\to\infty} \sum_{k=1}^{n} a_{n,k} = 1.$

Hence, we have established the necessity of three of these conditions.

Take

$$x_1 = x_2 = x_3 = \cdots = 1$$

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* Hurwitz: *Report*, p. 19.

and keep (y_n) arbitrary. Substituting in 18) we have

$$z_n = \sum_{k=1}^{n} a_{n,k} y_{n-k+1}$$

$$= \sum_{k=1}^{n} a_{n,n-k+1} y_k.$$

This is a regular transformation; hence in addition to b′) and c′)

$$\lim_{n\to\infty} a_{n,n-k+1} = 0, \qquad \text{for each } k.$$

It remains to establish the sufficiency of these conditions. Write

19) $$z_n = \sum_{k=1}^{n} b_{n,k} x_k,$$

where

$$b_{n,k} = a_{n,k} y_{n-k+1}$$

From condition a) we see that

20) $$\lim_{n\to\infty} b_{n,k} = 0, \qquad \text{for a fixed } k.$$

By condition c) and the fact that, for all n, $|\, y_n \,| \leq Y$, we have

21) $$\sum_{k=1}^{n} |\, b_{n,k} \,| = \sum_{k=1}^{n} |\, a_{n,k} \,|\ |\, y_{n-k+1} \,| < Y \cdot c.$$

We now write

$$\sum_{k=1}^{n} b_{n,k} = \sum_{k=1}^{n} a_{n,k} y_{n-k+1}$$

$$= \sum_{k=1}^{n} a_{n,n-k+1} y_k$$

Since the transformation having the matrix $(a_{n,n-k+1})$ is a regular transformation, we have

22) $$\lim_{n\to\infty} \sum_{k=1}^{n} b_{n,k} = y$$

Choose p, such that given $\epsilon > 0$,

$$|\, x_k - x \,| < \epsilon/2Yc, \qquad k > p.$$

From 19) we may write

$$z_n = \sum_{k=1}^{p} b_{n,k} x_k + \sum_{k=p+1}^{n} b_{n,k} x_k,$$

23) $$|\, z_n - x \sum_{k=1}^{n} b_{n,k} \,| \leq \sum_{k=1}^{p} |\, b_{n,k} \,|\ |\, x_k - x \,| + \sum_{k=p+1}^{n} |\, b_{n,k} \,|\ |\, x_k - x \,|$$

For the fixed number p

$$| x_k - x | < M, \qquad k \leq p$$

Substituting these results in 23), we have

$$| z_n - x \sum_{k=1}^{n} b_{n,k} | \leq M \sum_{k=1}^{p} | b_{n,k} | + \epsilon/2Yc \sum_{k=p+1}^{n} b_{n,k}.$$

By 20), given $\epsilon > 0$, we may choose $N > p$, so that

$$| b_{n,k} | < \epsilon/2pM, \qquad n > N.$$

Thus we have, for $n > N$,

$$| z_n - x \sum_{k=1}^{n} b_{n,k} | \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

By 22) we see that

$$\lim_{n \to \infty} z_n = xy.$$

The matrix $\mathcal{A}$, which we have used to give a bilinear transformation 18) of two sequences, may be considered also as giving in the usual way a transformation of a single sequence

$$y_n = \sum_{k=1}^{n} a_{n,k} x_k;$$

we may then speak of the algebraic properties of $\mathcal{A}$ with their usual meaning for transformations of this form.

A transformation of type $\mathcal{A}$ is pemutable with the fisrt Hölder mean, M, provided the matrix of $\mathcal{A}$ is permutable with the matrix of M.

LEMMA 2. *A necessary and sufficient condition that*

$$\mathcal{A}[E_r(x_n), E_r(y_n)] = E_r \mathcal{A}(x\,y).$$

is that the transformation $\mathcal{A}$ be permutable with M.

To show that this condition is necessary, we take the sequence

$$y_1 = y_2 = y_3 = \cdots = 1.$$

thus we have

24) $$\mathcal{A}[E_r(x_n), 1] = E_r \mathcal{A}(x_n, 1).$$

Define

25) $$\mathcal{A}(x_n, 1) = A(x_n),$$

where $A(x_n)$ is an ordinary sequence-to-sequence transformation.

Write Euler's Transformation *

$$26) \qquad y_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}(n-1)!}{(n-k)!(k-1)!} \quad x_k,$$

which we shall call Δ. It satisfies the condition

$$\Delta^2 = I,$$

where I is the identity, as is evident by an actual repetition of 26).

Form the transformation $\Delta E_r \Delta$ and designate it by E'_r, similarly $M' = \Delta M \Delta$. The transformation E_r is permutable † with M, consequently

$$E'_r M' = M' E_r'$$

A necessary and sufficient condition that a transformation be permutable with M' is that it be a multiplication, i. e. of the form ‡

$$y_n = f_n x_n.$$

Hence, E'_r is a multiplication and may be expressed in the form

$$y_n = \frac{1}{r^{n-1}} \quad x_n.$$

Form the transformation $A' = \Delta A \Delta$, and denote the elements of the matrix of A' by $(a'_{n,k})$. Applying $A'E'_r$ to (x_n), we have

$$27) \qquad \omega_n = \sum_{k=1}^{n} a'_{n,k} \quad \frac{1}{r^{k-1}} \quad x_k,$$

applying $E'_r A'$ to (x_n), we obtain

$$28) \qquad \Omega_n = \sum_{k=1}^{n} \frac{1}{r^{n-1}} \quad a'_{n,k} x_k.$$

In order that $\omega_n = \Omega_n$, for all (x), we see from 27) and 28) that $a'_{n,k} = 0$, when $k \neq n$.

Hence, A' is a multiplication and we have at once that A is permutable with M. From 25) we see that the matrix of $\mathcal{A}$ is the same as the matrix of A, then $\mathcal{A}$ is permutable with M.

It now remains to prove the sufficiency of this condition. All trans-

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* Bromwich: *An Introduction to the Theory of Infinite Series* (1908), p. 303.

† Hurwitz and Silverman: "On the Consistency and Equivalence of Certain Definitions of Summability," *Transactions of American Mathematical Society*, Vol. 18, p. 6. This paper will be referred to as Hurwitz and Silverman.

‡ Hurwitz and Silverman, p. 6.

formations permutable with M may be expressed in the following form *

$$y_n = \sum_{k=1}^{n} \sum_{h=k}^{n} \frac{(-1)^{h-k}(n-1)!}{(n-h)!(h-k)!(k-1)!} f_h x_h.$$

$\mathcal{A}$ is in this class, so we may write

$$a_{n,k} = \sum_{h=k}^{n} \frac{(-1)^{h-k}(n-1)!}{(n-h)!(h-k)!(k-1)!} f_h.$$

In 18) replace (x_n) by $E_r(x_n)$ and (y_n) by $E_r(y_n)$, then if we write

$$Z_n = \mathcal{A}[E_r(x_n),\ E_r(y_n)]$$

we have

$$Z_n = \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{(-1)^{h-k}(n-1)!}{(n-h)!(h-k)!(k-1)!} f_h \sum_{l=1}^{k} \frac{(k-1)!(r-1)^{k-l}}{(k-l)!(l-1)!\,r^{k-1}} x_l$$

$$\times \sum_{q=1}^{h-k+1} \frac{(h-k)!}{(h-k-q+1)!(q-1)!} \frac{(r-1)^{h-k-q+1}}{r^{h-k}} y_q,$$

$$29)\quad Z_n = \sum_{h=1}^{n} \sum_{l=1}^{h} \sum_{q=1}^{h-l+1} \sum_{k=l}^{h-q+1} \frac{(n-1)!(-1)^{h-k} f_h}{(n-h)!(k-l)!(l-1)!(h-k-q+1)!(q-1)!} \frac{(r-1)^{h-l-q+1}}{r^{h-1}} x_l x_q.$$

We find

$$\sum_{k=l}^{h-q+1} \frac{(-1)^k}{(k-l)!(h-k-q+1)!} = \begin{cases} 0, & h+1 \neq q+l, \\ (-1)^l & h+1 = q+l. \end{cases}$$

Substituting this result in 29), we obtain

$$30)\quad Z_n = \sum_{h=1}^{n} \sum_{l=1}^{h} \frac{(-1)^{h-l}(n-1)!}{(n-h)!(l-1)!(h-l)!} f_h \frac{1}{r^{h-1}} x_l y_{h-l+1}.$$

Applying E_r to (z_n) and writing

$$W_n = E_r \mathcal{A}(x_n, y_n),$$

we have

$$W_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}}$$

$$\times \sum_{h=1}^{k} \sum_{l=1}^{h} \frac{(k-1)!(-1)^{h-l}}{(k-h)!(h-l)!(l-1)!} f_h x_l y_{h-l+1},$$

$$31)\quad W_n = \sum_{l=1}^{n} \sum_{h=l}^{n} \sum_{k=h}^{n} \frac{(n-1)!(-1)^{h-l} f_h}{(n-k)!(k-h)!(h-l)!(l-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} x_l y_{h-l+1}$$

we find that

$$\sum_{k=h}^{n} \frac{1}{(n-k)!(k-h)!} \frac{1}{(r-1)^k} = \frac{1}{(n-h)!} \frac{1}{(r-1)^h} \left[1 + \frac{1}{r-1}\right]^{n-h}$$

$$= \frac{1}{(n-h)!} \frac{r^{n-h}}{(r-1)^n}$$

* Hurwitz and Silverman, p. 7.

When we substitute this result in 31), we see that

$$32) \qquad W_n = \sum_{l=1}^{n} \sum_{h=l}^{n} \frac{(n-1)!(-1)^{h-l}}{(n-h)!(h-l)!(l-1)!} f_h \frac{1}{r^{h-1}} x_l y_{h-l+1};$$

By comparing results of 30) and 32), we find that

$$W_n = Z_n,$$

or

$$\mathcal{A}[E_r(x_n), E_r(y_n)] = E_r \mathcal{A}(x_n y_n)$$

as we wished to prove.

Take the two sequences (x_n) and (y_n) and the infinite matrix

$$(m_{nk}) : \left\| \begin{array}{cccccccccc} 1/1, & 0, & 0, & 0, & . & . & . & . & . \\ 1/2, & 1/2, & 0, & 0, & . & . & . & . & . \\ 1/3, & 1/3, & 1/3, & 0, & 0, & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 1/n, & 1/n, & 1/n, & . & . & 1/n, & 0 & . & . \\ . & . & . & . & . & . & . & . & . \end{array} \right\|$$

Then the transformation

$$33) \qquad z_n = \sum_{k=1}^{n} m_{nk} x_k y_{n-k+1}$$

defines a sequence (z_n). We indicate the transformation by $\mathcal{M}$, and the relation by

$$(z_n) = \mathcal{M}(x_n y_n).$$

THEOREM VIII. *If $\sum u_n$ is summable E_r to the value σ and $\sum u_n$ is summable E_r to the value τ, then the Cauchy product $\sum w_n$ is summable ME_r to the value $\sigma\tau$.*

The two sequences formed from the two series are, respectively:

$$U_n = u_1 + u_2 + \cdots + u_n,$$
$$V_n = v_1 + v_2 + \cdots + v_n.$$

We shall now form the Cauchy product of the two series, where

$$w_n = u_1 v_n + u_2 v_{n-1} + \cdots + u_n v_1.$$

The sequence $W_1, W_2, \cdots$, is formed where

$$W_n = w_1 + w_2 + \cdots + w_n$$
$$= u_1 V_n + u_2 V_{n-1} + \cdots + u_n V_1,$$

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and

$$34) \qquad \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} U_i V_{n-i+1}.$$

Applying M and then E_r to (W_n), we have

$$35) \qquad y_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} \frac{1}{k} \sum_{h=1}^{k} W_k.$$

We shall now apply $\mathscr{M}$ to (U_n) and (V_n), and by 33) we obtain

$$z_n = 1/n \sum_{k=1}^{n} U_k V_{n-k+1}.$$

Applying E_r to (z_n), we have

$$36) \qquad z_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} \frac{1}{k} \sum_{h=1}^{k} U_h W_{k-h+1},$$

or

$$37) \qquad Z_n = E_r \mathscr{M}(U_n, V_n).$$

From 34), 35), 36), we see that

$$y_n = Z_n.$$

If now we show that $\lim_{n\to\infty} Z_n$ exists and equals $\sigma\tau$, our theorem is proved.

By Lemma 2, we have

$$38) \qquad E_r \mathscr{M}(U_n V_n) = \mathscr{M}[E_r(U_n), E_r(V_n)].$$

Let

$$\sigma_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} U_k,$$

$$\tau_n = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!} \frac{(r-1)^{n-k}}{r^{n-1}} V_k.$$

As $\sum u_n$, $\sum v_n$ are summable E_r,

$$\lim_{n\to\infty} \sigma_n = \sigma, \qquad \lim_{n\to\infty} \tau_n = \tau.$$

We have

$$39) \qquad \sigma_k = \sum_{p=1}^{k} \frac{(k-1)!}{(k-p)!(p-1)!} \frac{(r-1)^{k-p}}{r^{k-1}} U_p$$

similarly

$$40) \qquad \tau_{n-k+1} = \sum_{p=1}^{n-k+1} \frac{(n-k)!}{(n-k-p+1)!(p-1)!} \frac{(r-1)^{n-p-k+1}}{r^{n-p}} V_p.$$

From 37) and 38) we have

$$Z_n = \mathscr{M}[E_r(U_n), E_r(V_n)],$$

then substituting the values as given by 39) and 40) we find

$$Z_n = 1/n \sum_{k=1}^{n} \sigma_k \tau_{n-k+1}.$$

Obviously the conditions of Lemma 1 are satisfied and we state that

$$\lim_{n\to\infty} Z_n = \sigma\tau.$$

COROLLARY 1. *If $\sum u_n$ is summable E_r to the value σ and $\sum v_n$ is summable E_s to the value τ, then if $r > s$, the Cauchy product $\sum w_n$ is summable ME_r to the value $\sigma\tau$.*

This follows at once since the exponential mean becomes more inclusive as r increases.*

COROLLARY 2. *If $\sum u_n$, $\sum v_n$, and the Cauchy product $\sum w_n$ are summable † E_r to the values σ, τ, ω respectively, then $\omega = \sigma\tau$.*

If C_p denotes the Cesaro mean of order p then †

41) $$C_{\mu+\nu+1}(W_n) = \mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

where $\mathfrak{D}$ is the transformation

$$Z_n = \sum_{k=1}^{n} d_{n,k} x_k y_k,$$

with

$$d_{n,k} = \frac{(\mu+\nu+1)\,!\,(n-1)\,!\,(n+k-1)\,!\,(n-k+\nu)\,!}{\mu\,!\,\nu\,!\,(n-k)\,!\,(k-1)\,!\,(\mu+\nu+n)\,!}.$$

This transformation satisfies the conditions of Lemma 1; this consideration furnishes the usual proof ‡ of the theorem: *If $\sum u_n$ is summable C_μ to the value σ and $\sum v_n$ is summable to value τ, then the Cauchy product $\sum w_n$ is summable $C_{\mu+\nu+1}$ to value $\sigma\tau$.*

We proceed to generalize this result.

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* Hurwitz: *Report*, p. 27.

† Knopp: "Über das Eulersche Summierungsverfahren," *Mathematische Zeitschrift*, Vol. 15 (1922-23), p. 226.

Knopp gives a definition formed by applying E_2 successively p times. He proves for this definition the results stated in this corollary. But since $(E_2)^p = E_{2p}$, his result is the special case in which r, instead of being any number greater than 1, is restricted to be an even integer.

‡ Chapman: "Non-Integral Orders of Summability of Series and Integrals," *Proceedings of London Mathematical Society*, Series 2, Vol. 9 (1910), p. 369.

THEOREM IX. *If* $\sum u_n$ *is summable* $C_\mu E_r$ *to the value* σ *and* $\sum v_n$ *is summable* $C_\nu E_r$ *to the value* τ, *then their Cauchy product is summable* $C_{\mu+\nu+1} E_r$ *to the value* $\sigma\tau$.

Since the transformation $\mathfrak{D}$ is permutable with M, we have

42) $$\mathfrak{D}[C_\mu E_r(U_n), C_\nu E_r(V_n)] = E_r \mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

multiplying each side of 41) on the left by E_r, we obtain

43) $$E_r C_{\mu+\nu+1}(W_n) = E_r \mathfrak{D}[C_\mu(U_n), C_\nu(V_n)],$$

From 42) and 43) we have

44) $$E_r C_{\mu+\nu+1}(W_n) = \mathfrak{D}[C_\mu E_r(U_n), C_\nu E_r(V_n)].$$

Thus we see from 44) that (W_n) is summable $E_r C_{\mu+\nu+1}$ to the value $\sigma\tau$.

COROLLARY 1. *If* $\sum u_n$ *is summable* $C_\mu E_r$ *to the value* σ, *and* $\sum v_n$ *is summable* $C_\nu E_s$ *to the value* τ, *then if* $r > s$, *the Cauchy product is summable* $C_{\mu+\nu+1} E_r$ *to the value* $\sigma\tau$.

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